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On Some L^p Inequalities in Best Approximation Theory

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1. INTRODUCTION

Throughout this paper p will denote a number greater than or equal to 2 and λ will always stand for $1-\mu$. As shown in [1], for $0<\mu\leqslant 1/2$, $\lambda=1-\mu$, the function $h(x)=\lambda x^{p-1}-\mu-(\lambda x-\mu)^{p-1}$ is strictly increasing in the interval $\mu/\lambda\leqslant x\leqslant 1$, $h(\mu/\lambda)<0$, and $h(1)>0$; thus $h(x)=0$ has a unique solution $x(\mu)$. (Note that $x(1/2)=1$.) $x(\mu)$ is strictly increasing in μ and $x(0+)$ is the unique solution of the equation

$$(p-2)\,x^{p-1}+(p-1)\,x^{p-2}=1 \tag{1.1}$$

in the interval $0\leqslant x\leqslant 1$.

In [1], we proved the inequality

$$\|\lambda x+\mu y\|^p+\lambda\mu f(\mu)\,\|x-y\|^p\leqslant\lambda\,\|x\|^p+\mu\,\|y\|^p, \tag{1.2}$$

where $0\leqslant\mu\leqslant 1$, $\lambda=1-\mu$, $p\geqslant 2$, x, $y\in L^p$, and $f(\mu)$ is a function of μ (and hence of λ) defined by

$$f(\mu)=\frac{1+x(\mu\wedge\lambda)^{p-1}}{(1+x(\mu\wedge\lambda))^{p-1}}. \tag{1.3}$$

Moreover, the function $g(\mu)=\lambda\mu f(\mu)$ is best possible; thus it follows that for any function $k(\mu)$, the inequality

$$\|\lambda x+\mu y\|^p+k(\mu)\,\|x-y\|^p\leqslant\lambda\,\|x\|^p+\mu(\,y\|^p \tag{1.4}$$

is valid for all x, $y\in L^p$ if and only if $k(\mu)\leqslant g(\mu)$. It was also proved in [1] that $f(\mu)$ is strictly decreasing in μ for $0<\mu\leqslant 1/2$ (see the proof of Lemma 4 in [1]). Thus, we have

$$2^{2-p}=f(1/2)\leqslant f(\mu)\leqslant f(0+)\equiv c_p. \tag{1.5}$$

Note that the c_p defined here is the same as the c_p defined in [2].

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In this paper, we show that some L^p inequalities proved in Smarzewski [2, 3] follow readily from the preceding remarks. We also correct a proof of an inequality in [2] and show that the constant in that inequality is the best possible. It then follows that a fixed point theorem in [2] is not more general the a previous result of the present author [1]. Finally, we prove that our inequalities are also valid for complex-valued functions.

2. L^p INEQUALITIES

From (1.5), we have

$$2^{2-p}(\lambda\mu^p + \mu\lambda^p) \leqslant f(\mu)\, \lambda\mu = g(\mu).$$

Thus the inequality

$$t\,|u+v|^p + (1-t)\,|u|^p + |u+tv|^p \geqslant 2^{2-p}(t(1-t)^p + (1-t)\,t^p)\,|v|^p$$

in Lemma 4.2 of [3] follows immediately from (1.4).

It also follows from (1.2) that

$$\frac{\|x+\mu(y-x)\|^p - \|x\|^p}{\mu} \leqslant \|y\|^p - \|x\|^p - (1-\mu)\,f(\mu)\,\|x-y\|^p.$$

By letting $\mu \to 0$, we get, with $\Phi(x) = \|x\|^p$,

$$\begin{aligned} D\Phi(x, y-x) &= \langle p\,|x(t)|^{p-1} \operatorname{sgn} x(t), y(t)-x(t)\rangle \\ &\leqslant \|y\|^p - \|x\|^p - c_p\,\|x-y\|^p \end{aligned}$$

which is the inequality (3.10) in [2]. In particular, if x, y are real numbers, we have

$$p\,|x|^{p-2}x(y-x) \leqslant |y|^p - |x|^p - c_p\,|x-y|^p \tag{2.1}$$

which is the inequality in Lemma 3.1 of [2].

If we let $s(\mu) = x(\mu)/(1+x(\mu))$, then for $0 < \mu \leqslant 1/2$, $s(\mu)$ is the unique solution of the equation

$$\lambda s^{p-1} - \mu(1-s)^{p-1} - (s-\mu)^{p-1} = 0$$

in the interval $\mu \leqslant s \leqslant 1/2$. Since $x(\mu)$ is strictly increasing, $s(\mu)$ must also be strictly increasing. Thus the proof of the inequality

$$|u+t(v-u)|^p - |u|^p \leqslant t(|v|^p - |u|^p) - c_p w(t)\,|v-u|^p, \tag{2.2}$$

where $w(t) = t(1-t)^p + (1-t)\,t^p$, as given in [2] is incorrect.

Obviously, (2.2) is equivalent to the inequality

$$|\lambda x+\mu y|^p+c_p w(\mu)|x-y|^p \leqslant \lambda|x|^p+\mu|y|^p$$

which, in turn, is equivalent to $c_p w(\mu)\leqslant g(\mu)$ by (1.4), or to

$$f(0+)=c_p\leqslant \frac{s(\mu)^{p-1}+(1-s(\mu))^{p-1}}{\mu^{p-1}+\lambda^{p-1}}=\frac{f(\mu)}{\mu^{p-1}+\lambda^{p-1}}. \tag{2.3}$$

Since both $f(\mu)$ and $\mu^{p-1}+\lambda^{p-1}$ are strictly decreasing on $0<\mu\leqslant 1/2$, inequality (2.3) is not obvious and a direct proof may be difficult.

THEOREM 2.1. *For $p\geqslant 2$, $0\leqslant\mu\leqslant 1$, and $\lambda=1-\mu$, the inequality*

$$\|\lambda x+\mu y\|^p+c_p\lambda\mu(\lambda^{p-1}+\mu^{p-1})\|x-y\|^p\leqslant\lambda\|x\|^p+\mu\|y\|^p \tag{2.4}$$

is valid for all x, $y\in L^p$. Moreover, the constant c_p is best possible.

Proof. It suffices to prove (2.4) for real numbers x and y. We may assume that $|x|\leqslant|y|$. It is then easy to see that the best constant for (2.4) to be valid is

$$c\equiv\inf_{-1\leqslant x<1}\ \inf_{0<\mu\leqslant 1}\frac{\mu+\lambda|x|^p-|\mu+\lambda x|^p}{W(\mu)(1-x)^p},$$

where $W(\mu)=\lambda\mu(\mu^{p-1}+\lambda^{p-1})$. For a fixed x, $-1\leqslant x<1$, let

$$r(\mu)=\frac{\mu+\lambda|x|^p-|\mu+\lambda x|^p}{W(\mu)},\qquad 0<\mu\leqslant 1.$$

Since $\lim_{\mu\to 0}(W(\mu)/\mu)=1$, we have $r(0+)=1-|x|^p-p|x|^{p-1}\operatorname{sgn}(x)(1-x)$. We shall prove that $\inf_{0<\mu\leqslant 1}r(\mu)=r(0+)$, which is equivalent to

$$\mu+\lambda|x|^p-|\mu+\lambda x|^p-W(\mu)(1-|x|^p-p|x|^{p-1}\operatorname{sgn}(x)(1-x))\geqslant 0. \tag{2.5}$$

Since $\lambda+\mu|x|^p-|\lambda+\mu x|^p\geqslant\mu+\lambda|x|^p-|\mu+\lambda x|^p$ for $0\leqslant\mu\leqslant 1/2$ (cf. proof of Lemma 3 in [1]) and $W(\mu)=W(\lambda)$, we may assume that $0\leqslant\mu\leqslant 1/2$, or equivalently, $\lambda\geqslant\mu$ in (2.5).

If x is negative, then upon replacing x by $-x$ and dividing by x^p, (2.5) is equivalent to

$$F(y)\equiv\mu y^p+\lambda-|\mu y-\lambda|^p-W(\mu)(p-1+py+y^p)\geqslant 0$$

for $y\geqslant 1$, where F is regarded as a function of y. We have

$$F'(y)=\mu p[y^{p-1}-|\mu y-\lambda|^{p-1}\operatorname{sgn}(\mu y-\lambda)-\lambda(\mu^{p-1}+\lambda^{p-1})(y^{p-1}+1)]$$

and

$$F''(y) = \mu p(p-1)[(1-\lambda(\mu^{p-1}+\lambda^{p-1}))\, y^{p-2} - \mu|\mu y - \lambda|^{p-2}].$$

Since $1-\lambda(\mu^{p-1}+\lambda^{p-1}) \geqslant \mu$ and $y \geqslant |\mu y - \lambda|$, we have $F''(y) \geqslant 0$. Thus it suffices to show that $F(1) \geqslant 0$ and $F'(1) \geqslant 0$.

By elementary calculus, one can prove $(1+\zeta)^{p-1} + (1-\zeta)^{p-1} \geqslant 2$ and then $(1+\zeta)^p - (1-\zeta)^p \geqslant 2p\zeta$ for $0 \leqslant \zeta \leqslant 1$; thus

$$\begin{aligned} F(1) &= (\lambda+\mu)^p - (\lambda-\mu)^p - 2pW(\mu) \\ &\geqslant 2p\lambda^{p-1}\mu - 2pW(\mu) \\ &= 2p\lambda\mu^2(\lambda^{p-2} - \mu^{p-2}) \geqslant 0. \end{aligned}$$

It remains to show that

$$F'(1) = 1 + (\lambda-\mu)^{p-1} - 2\lambda(\mu^{p-1} + \lambda^{p-1}) \geqslant 0$$

or, equivalently (with $v = \lambda - \mu$),

$$2^r(1+v^r) - (1+v)((1+v)^r + (1-v)^r) \geqslant 0$$

for $0 \leqslant v \leqslant 1$, where $r = p-1$. By the Hölder inequality we have $(1+v)^r \leqslant 2^{r-1}(1+v^r)$; thus

$$\begin{aligned} &2^r(1+v^r) - (1+v)((1+v)^r + (1-v)^r) \\ &\quad \geqslant 2^r(1+v^r) + (1+v)(2^{r-1}(1+v^r) + (1-v)^r) \\ &\quad = (1-v)\{2^{r-1}(1+v^r) - (1+v)(1-v)^{r-1}\} \\ &\quad \geqslant (1-v)\{2^{r-1}(1+v^r) - (1-v)^r\} \geqslant 0. \end{aligned}$$

The case for $x \geqslant 0$ can be proved similarly. Thus

$$c = \inf_{-1 \leqslant x < 1} \frac{1 - |x|^p - p|x|^{p-1}\operatorname{sgn}(x)(1-x)}{(1-x)^p} \geqslant c_p$$

by (2.1). On the other hand, since $g(\mu)$ as in Section 1 is best possible we must have $cW(\mu) \leqslant g(\mu)$, or

$$c(\mu^{p-1} + \lambda^{p-1}) \leqslant f(\mu).$$

Letting $\mu \to 0$, we get $c \leqslant c_p$. Hence $c = c_p$. This completes the proof. ∎

Remark 2.1. Since c_p is best possible in (2.4), we see that Theorem 4.2 in [2] for $p \geqslant 2$ is not more general than a previous theorem (Theorem 1 in [1]) of the present author.

To see that inequality (1.2) is also valid for complex-valued functions, we first prove the following lemma.

LEMMA 2.1. *For fixed μ and r in $[0, 1]$, μ and r not both zeros, the function*

$$A(p) = \frac{(\mu + \lambda r)^p}{\mu + \lambda r^p}$$

is decreasing for $p \geqslant 1$. (Recall that $\lambda = 1 - \mu$.)

Proof. Let $B(p) = \ln A(p) = p \ln(\mu + \lambda r) - \ln(\mu + \lambda r^p)$. Then $(\mu + \lambda r^p) B'(p) = (\mu + \lambda r^p) \ln(\mu + \lambda r) - \lambda r^p \ln(r) \equiv C(\mu)$. Direct computation shows that $C''(\mu)$ is nonnegative. Since $C(0) = C(1) = 0$, we have $C(\mu) \leqslant 0$ and hence $B'(p) \leqslant 0$. This clearly completes the proof. ▮

THEOREM 2.2. *Inequality* (1.2) *is also valid for complex functions x and y.*

Proof. We need to prove that

$$\inf_{|z| \leqslant 1, z \neq 1, z \in \mathbb{C}} \frac{\mu + \lambda |z|^p - |\mu + \lambda z|^p}{|1 - z|^p} = \inf_{-1 \leqslant x < 1} \frac{\mu + \lambda |x|^p - |\mu + \lambda x|^p}{(1 - x)^p}.$$

Thus we may assume that $y = 1$ and $x = re^{i\theta}$ in (1.2), where $0 \leqslant r \leqslant 1$. Writing ζ for $\cos\theta$, we see that it suffices to prove that

$$q(\zeta) = \frac{\mu + \lambda r^p - (\mu^2 + 2\lambda\mu r\zeta + \lambda^2 r^2)^{p/2}}{(1 - 2r\zeta + r^2)^{p/2}}$$

as a function of ζ, $-1 \leqslant \zeta \leqslant 1$, attains its minimum at $\zeta = -1$. We have

$$q'(\zeta) = \frac{rp}{(1 - 2r\zeta + r^2)^{2/p + 1}} \mu(\zeta),$$

where $u(\zeta) = \mu + \lambda r^p - (\mu^2 + 2\mu\lambda r\zeta + \lambda^2 r^2)^{p/2 - 1}(\mu + \lambda r^2)$. Since $u(\zeta)$ is decreasing in ζ and $u(1) \geqslant 0$ by Lemma 2.1, we have $u(\zeta) \geqslant 0$ and hence $q'(\zeta) \geqslant 0$. This clearly completes the proof. ▮

Similarly, we have

THEOREM 2.3. *Inequality* (2.4) *is also valid for complex functions x and y.*

Proof. From the proofs of Theorem 2.1 and 2.2, we have

$$c_p = \inf_{-1 \leqslant x < 1} \lim_{0 < \mu \leqslant 1} \frac{\mu + \lambda |x|^p - |\mu + \lambda x|^p}{W(\mu)(1-x)^p}$$

$$= \inf_{0 < \mu \leqslant 1} \inf_{-1 \leqslant x < 1} \frac{\mu + \lambda |x|^p - |\mu + \lambda x|^p}{W(\mu)(1-x)^p}$$

$$= \inf_{0 < \mu \leqslant 1} \inf_{|z| \leqslant 1, z \neq 1, z \in \mathbb{C}} \frac{\mu + \lambda |z|^p - |\mu + \lambda z|^p}{W(\mu)|1-z|^p}$$

and (2.4) follows. ∎

Remark 2.2. As in the real case, inequality (2.1) for complex x, y follows immediately from Theorem 2.2.

For $1 < p < 2$, inequalities (1.2) and (2.4) are reversed. The proofs of these are similar to the proofs of Theorem 3 in [1] and Theorem 1 above, respectively. Thus, we omit the proof of the following theorem.

THEOREM 2.4. *For $1 < p \leqslant 2$, $0 \leqslant \mu \leqslant 1$, and $\lambda = 1 - \mu$, the inequalities*

$$\|\lambda x + \mu y\|^p + \lambda\mu f(\mu) \|x - y\|^p \geqslant \lambda \|x\|^p + \mu \|y\|^p$$

$$\|\lambda x + \mu y\|^p + c_p \lambda\mu(\lambda^{p-1} + \mu^{p-1}) \|x - y\|^p \geqslant \lambda \|x\|^p + \mu \|y\|^p$$

are valid for all x, $y \in L^p$(the complex L^p space). Moreover, the function $\lambda\mu f(\mu)$ and the constant c_p are best possible.

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